MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes IX

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Question

Evaluate the following integrals:

(Q1) $\int \left| x^2 + x - 6 \right| dx$

(Q2) $\int \frac{1}{\sin^2 x \cos^2 x} dx$

(Q3) $\int \left(1 + \tan^6 x\right) dx$

(Q4) $\int \frac{x^{2n-1}}{x^n + 1} dx \qquad (n \in \mathbb{N})$

(Q5) $\int \sin^2 x dx$

(Q6) $\int x \tan^{-1} x dx$

(Q7) $\int e^{2x} \cos 3x dx$

(Q8) $\int \frac{1}{x^2 + x + 1} dx$

(Q9) $\int \frac{1}{x^2 - 2x \cos \alpha + 1} dx \qquad \left(0 < \alpha < \pi\right)$

(Q10) $\int \frac{1}{x^2 - 100} dx$

(Q11) $\int \frac{1}{x^3 + 1} dx$

Answer

(A1) Note $\left|x^2+x-6\right| = \begin{cases} x^2+x-6, & \text{if } x^2+x-6 \geq 0 \\ -x^2-x+6, & \text{if } x^2+x-6 < 0 \end{cases} = \begin{cases} x^2+x-6, & \text{if } x \geq 2 \\ -x^2-x+6, & \text{if } -3 \leq x < 2 \\ x^2+x-6, & \text{if } x < -3 \end{cases}.$

Then $\displaystyle\int \left|x^2+x-6\right| dx = \begin{cases} \frac{x^3}{3}+\frac{x^2}{2}-6x+C, & \text{if } x \geq 2 \\ -\frac{x^3}{3}-\frac{x^2}{2}+6x+C', & \text{if } -3 \leq x < 2 \\ \frac{x^3}{3}+\frac{x^2}{2}-6x+C'', & \text{if } x < -3 \end{cases}.$

Note that $\displaystyle\int \left|x^2+x-6\right| dx$ is continuous, hence

$$\begin{aligned} \lim_{x\to 2^-} \int \left|x^2+x-6\right| dx &= \lim_{x\to 2^+} \int \left|x^2+x-6\right| dx \\ \lim_{x\to 2^-} \left(-\frac{x^3}{3}-\frac{x^2}{2}+6x+C'\right) &= \lim_{x\to 2^+} \left(\frac{x^3}{3}+\frac{x^2}{2}-6x+C\right) \\ \frac{22}{3}+C' &= -\frac{22}{3}+C \\ C' &= -\frac{44}{3}+C \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to -3^-} \int \left|x^2+x-6\right| dx &= \lim_{x\to -3^+} \int \left|x^2+x-6\right| dx \\ \lim_{x\to -3^-} \left(\frac{x^3}{3}+\frac{x^2}{2}-6x+C''\right) &= \lim_{x\to -3^+} \left(-\frac{x^3}{3}-\frac{x^2}{2}+6x+C'\right) \\ \frac{27}{2}+C'' &= -\frac{9}{2}+C' \\ C'' &= -18+C' = -18-\frac{44}{3}+C = -\frac{98}{3}+C. \end{aligned}$$

Hence, $\displaystyle\int \left|x^2+x-6\right| dx = \begin{cases} \frac{x^3}{3}+\frac{x^2}{2}-6x+C, & \text{if } x \geq 2 \\ -\frac{x^3}{3}-\frac{x^2}{2}+6x-\frac{44}{3}+C, & \text{if } -3 \leq x < 2 \\ \frac{x^3}{3}+\frac{x^2}{2}-6x-\frac{98}{3}+C, & \text{if } x < -3 \end{cases}.$

(A2)

$$\begin{aligned} \int \frac{1}{\sin^2 x \cos^2 x} dx &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \left(\frac{1}{\sin^2 x}+\frac{1}{\cos^2 x}\right) dx \\ &= \int \left(\csc^2 x + \sec^2 x\right) dx \\ &= \tan x - \cot x + C \end{aligned}$$

(A3)

$$\begin{aligned}\int \left(1+\tan^6 x\right) dx &= \int \left(1+\tan^2 x\right)\left(1-\tan^2 x+\tan^4 x\right) dx \\ &= \int \sec^2 x \left(1-\tan^2 x+\tan^4 x\right) dx \\ &= \int \left(1-\tan^2 x+\tan^4 x\right) d\tan x \\ &= \tan x - \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C\end{aligned}$$

(A4) Using Long Division, we have

$$\begin{aligned}\int \frac{x^{2n-1}}{x^n+1} dx &= \int \left(x^{n-1} - \frac{x^{n-1}}{x^n+1}\right) dx \\ &= \frac{x^n}{n} - \frac{1}{n}\int \frac{1}{x^n+1} d\left(x^n+1\right) \\ &= \frac{x^n}{n} - \frac{\ln\left|x^n+1\right|}{n} + C\end{aligned}$$

(A5) There are two method:

(Method 1) $\int \sin^2 x dx = \frac{1}{2}\int \left(1-\cos 2x\right) dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$

(Method 2) Using integration by part, we have

$$\begin{aligned}\int \sin^2 x dx &= -\int \sin x d\cos x \\ &= -\sin x\cos x + \int \cos x d\sin x \\ &= -\sin x\cos x + \int \cos^2 x dx \\ &= -\sin x\cos x + \int \left(1-\sin^2 x\right) dx \\ &= -\sin x\cos x + x - \int \sin^2 x dx \\ 2\int \sin^2 x dx &= x - \sin x\cos x + C \\ \int \sin^2 x dx &= \frac{x-\sin x\cos x}{2} + C\end{aligned}$$

(A6)

$$\begin{aligned}\int x\tan^{-1} x dx &= \frac{1}{2}\int \tan^{-1} x dx^2 \\ &= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int x^2 d\tan^{-1} x \\ &= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2} dx \\ &= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1-\frac{1}{1+x^2}\right) dx \\ &= \frac{x^2}{2}\tan^{-1} - \frac{x}{2} + \frac{\tan^{-1} x}{2} + C\end{aligned}$$

(A7)

$$
\begin{aligned}
\int e^{2x}\cos 3x\,dx &= \frac{1}{2}\int \cos 3x\,de^{2x} \\
&= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\,dx \\
&= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}\int \sin 3x\,de^{2x} \\
&= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\,dx \\
\frac{13}{4}\int e^{2x}\cos 3x\,dx &= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x + C \\
\int e^{2x}\cos 3x\,dx &= \frac{e^{2x}}{13}\left(2\cos 3x + 3\sin 3x\right) + C
\end{aligned}
$$

(A8) Note that $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.

Let $x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$.

Then $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\tan^2\theta + \frac{3}{4} = \frac{3}{4}\sec^2\theta$,

and $dx = \frac{\sqrt{3}}{2}\sec^2\theta d\theta$. Then

$$
\begin{aligned}
\int \frac{1}{x^2 + x + 1}dx &= \int \frac{1}{\frac{3}{4}\sec^2\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta d\theta \\
&= \int \frac{2}{\sqrt{3}}d\theta \\
&= \frac{2}{\sqrt{3}}\theta + C \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)\right) + C \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) + C
\end{aligned}
$$

(A9) Note that $x^2 - 2x\cos\alpha + 1 = x^2 - 2x\cos\alpha + \cos^2\alpha + \sin^2\alpha = (x - \cos\alpha)^2 + \sin^2\alpha$.

Let $x - \cos\alpha = \sin\alpha\tan\theta$.

Then $(x - \cos\alpha)^2 + \sin^2\alpha = \sin^2\alpha\sec^2\theta$ and $dx = \sin\alpha\sec^2\theta d\theta$. Then

$$
\begin{aligned}
\int \frac{1}{x^2 - 2x\cos\alpha + 1}dx &= \int \frac{1}{\sin^2\alpha\sec^2\theta}\cdot\sin\alpha\sec^2\theta d\theta \\
&= \theta\csc\alpha + C \\
&= \csc\alpha\tan^{-1}(x\csc\alpha - \cot\alpha) + C
\end{aligned}
$$

(A10) Note that $x^2 - 100 = (x - 10)(x + 10)$.

Write $\frac{1}{x^2 - 100} = \frac{A}{x - 10} + \frac{B}{x + 10} = \frac{(A + B)x + (10A - 10B)}{x^2 - 100}$, we can have $A = \frac{1}{20}$ and $B = -\frac{1}{20}$.

$$\begin{aligned}\int \frac{1}{x^2 - 100} dx &= \frac{1}{20} \int \frac{dx}{x - 10} - \frac{1}{20} \int \frac{dx}{x + 10} \\ &= \frac{\ln|x - 10| - \ln|x + 10|}{20} + C \\ &= \frac{1}{20} \ln\left|\frac{x - 10}{x + 10}\right| + C\end{aligned}$$

(A11) Note that $x^3 + 1 = (x + 1)\left(x^2 - x + 1\right)$, write

$$\begin{aligned}\frac{1}{x^3 + 1} &= \frac{A}{x + 1} + \frac{Bx + C}{x^2 - x + 1} \\ &= \frac{(A + B)x^2 + (-A + B + C)x + (A + C)}{x^3 + 1}.\end{aligned}$$

Then $A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3}$. Note that

$$\begin{aligned}\int \frac{2 - x}{3\left(x^2 - x + 1\right)} dx &= -\frac{1}{6} \int \frac{2x - 1}{x^2 - x + 1} dx + \frac{1}{2} \int \frac{1}{x^2 - x + 1} dx \\ &= -\frac{1}{6} \int \frac{d\left(x^2 - x + 1\right)}{x^2 - x + 1} + \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} dx \\ &= -\frac{1}{6} \ln\left|x^2 - x + 1\right| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + C'.\end{aligned}$$

Then

$$\begin{aligned}\int \frac{1}{x^3 + 1} dx &= \int \frac{1}{3\left(x + 1\right)} dx + \int \frac{2 - x}{3\left(x^2 - x + 1\right)} dx \\ &= \frac{1}{3} \ln|x + 1| - \frac{1}{6} \ln\left|x^2 - x + 1\right| + \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) + C'.\end{aligned}$$